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Solution for 'Topics in complex analysis'

(24/09/2025)

H 3.1 (A convergence criterion)
Let $D \subset \mathbb{C}$ be a domain and let $f_n : D \to \mathbb{C}$ be a sequence of holomorphic functions that is locally uniformly bounded. Assume that there exists $z_0 \in D$ such that for each $k \in \mathbb{N} \cup \{0\}$ the limit

$$\lim_{n \to +\infty} f_n^{(k)}(z_0)$$

exists. Show that the whole sequence f_n converges locally uniformly to some $f : D \to \mathbb{C}$.

Solution H 3.1: Due to Montel's theorem it suffices to prove pointwise convergence[1]. As seen in the proof of Vitali's theorem (c.f. lecture or use the same idea as in the previous footnote), non-convergence at a point $z' \in D$ implies the existence of two holomorphic functions $h, g : D \to \mathbb{C}$ and two subsequences $f_{n_{k,1}}$ and $f_{n_{k,2}}$ such that

(i) $f_{n_{k,1}} \to h$ and $f_{n_{k,2}} \to g$ locally uniformly as $k \to +\infty$;

(ii) $h(z') \neq g(z')$.

Applying Theorem 1.5 we infer that for each $m \in \mathbb{N}$ we have

$$f_{n_{k,1}}^{(m)} \to h^{(m)} \quad \text{and} \quad f_{n_{k,2}}^{(m)} \to g^{(m)} \quad \text{locally uniformly as } k \to +\infty.$$

Our assumption implies that $h^{(m)}(z_0) = g^{(m)}(z_0)$ for all $m \in \mathbb{N}$. Thus by analyticity of h and g, the set $\{h = g\}$ contains a small ball $B_r(z_0)$, with $r > 0$. By the identity theorem we conclude that $h = g$, which yields a contradiction at z'. □

H 3.2 (From the exam in 2019)
Denote by $B_1(0) \subset \mathbb{C}$ the open unit disc. Define the family of functions

$$\mathcal{F} = \left\{ f : B_1(0) \to \mathbb{C},\ f \text{ holomorphic},\ f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |a_k| \leq 1 \quad \forall k \in \mathbb{N} \cup \{0\} \right\}.$$

[1] Indeed, suppose we have proven pointwise convergence to some function $f : D \to \mathbb{C}$ and assume by contradiction that the sequence f_n does not converge locally uniformly to f. Then there exists $\varepsilon > 0$, a subsequence f_{n_k}, and a compact set $K \subset D$ such that

$$\sup_{z \in K} |f_{n_k}(z) - f(z)| \geq \varepsilon \quad \text{for all } k \geq 1.$$

Along this subsequence we use Montel's theorem, so that a further subsequence $f_{n_{k,1}}$ converges locally uniformly to some holomorphic $g : D \to \mathbb{C}$. By the (assumed) pointwise convergence of f_n and uniqueness of analytic continuation, it follows that $f = g$, which contradicts the above estimate (as f_{n_k} cannot converge locally uniformly to f).

Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ be a sequence. Show that some subsequence of f_n converges locally uniformly to a function $f \in \mathcal{F}$.

Remark: The representation $f(z) = \sum_{k=0}^{\infty} a_k z^k$ holds for any holomorphic function $f : B_1(0) \to \mathbb{C}$.

Solution 3.2: Since $|a_k| \leq 1$, the triangle inequality implies

$$|f(z)| \leq \sum_{k=0}^{\infty} |a_k||z|^k \leq \sum_{k=0}^{\infty} |z|^k = \frac{1}{1-|z|}$$

for all $z \in B_1(0)$ and $f \in \mathcal{F}$. Hence for any compact subset $K \subset B_1(0)$ the sequence f_n is uniformly bounded on K. By Montel's theorem there exists a subsequence f_{n_j} and $f : B_1(0) \to \mathbb{C}$ such that $f_{n_j} \to f$ locally uniformly as $j \to +\infty$. Since each f_n is holomorphic it follows from Theorem 1.3 that f is also holomorphic. Thus it remains to show that the coefficients of its series representation are bounded in modulus by 1. Denote the coefficients of f_{n_j} by a_k^j and the coefficients of f by a_k. Then $a_k^j = f_{n_j}^{(k)}(0)/k!$ and similarly $a_k = f^{(k)}(0)/k!$. From Theorem 1.5 we deduce that $f_{n_j}^{(k)}(0) \to f^{(k)}(0)$ for all $k \in \mathbb{N} \cup \{0\}$. Hence recalling that $|a_k^j| \leq 1$ since $f_{n_j} \in \mathcal{F}$,

$$|a_k| = \frac{|f^{(k)}(0)|}{k!} = \lim_{j\to\infty} \frac{|f_{n_j}^{(k)}(0)|}{k!} = \lim_{j\to\infty} |a_k^j| \leq 1,$$

so $f \in \mathcal{F}$. □

H 3.3 (An extremal problem in the proof of the Riemann mapping theorem)
Let $D \subsetneq \mathbb{C}$ be a simply connected domain such that $0 \in D$. Show that there exists a holomorphic function $f : D \to \mathbb{C}$ that solves the extremal problem

$$s_0 = \sup\{|f'(0)| : \text{the function } f : D \to B_1(0) \text{ is holomorphic and injective with } f(0) = 0\}.$$

Hint: Consider a sequence of admissible functions $f_n : D \to B_1(0)$ such that $|f_n'(0)| \to s_0$. You may assume without proof that there exists at least one admissible function for the above optimization problem. Indeed, we will prove that later based on the fact that D is simply connected.

Solution H 3.3: Take a sequence $f_n : D \to B_1(0)$ of holomorphic functions such that each f_n is injective, $f_n(0) = 0$, and

$$\lim_{n\to+\infty} |f_n'(0)| = s_0.$$

(Such a sequence exists once we know that there exists any holomorphic function $f : D \to B_1(0)$ that is injective with $f(0) = 0$. This we will prove in a forthcoming lecture and you can assume.) If $s_0 = 0$, then any admissible function solves the extremal problem and we are done[2]. Hence assume that $s_0 > 0$. Since $f_n(D) \subset B_1(0)$ we know that $|f_n(z)| \leq 1$, so that f_n is (locally) uniformly bounded on D. By Montel's theorem we find a subsequence (which from now on we pass to) and a holomorphic function $f : D \to \mathbb{C}$ such that $f_n \to f$ locally uniformly on D. In particular, for any fixed $z \in D$ we have $|f(z)| = \lim_{n\to+\infty} |f_n(z)|$, which implies that $f : D \to \overline{B_1(0)}$ and $f(0) = 0$. Now assume for the moment that there exists $z_0 \in D$ such that $|f(z_0)| = 1$. Then by the maximum principle f would be constant, which contradicts the fact that $f(0) = 0$. Hence actually $f : D \to B_1(0)$. Moreover, due to Theorem 1.5 it holds that

$$|f'(0)| = \lim_{n\to+\infty} |f_n'(0)| = s_0.$$

[2]We will show later that an injective holomorphic function cannot have vanishing derivative in any point, so this case actually never occurs.

Thus f solves the extremal problem once we show that f is injective. Assume by contradiction that f is not injective, so that there exist $z_1, z_2 \in D$ such that $f(z_1) = f(z_2)$. Then the function $\widetilde{f}(z) = f(z) - f(z_1)$ has at least two zeros, while due to injectivity each $\widetilde{f}_n(z) := f_n(z) - f(z_1)$ has at most one zero. Since $\widetilde{f}_n \to \widetilde{f}$ locally uniformly, it follows[3] from Corollary 1.6 that $\widetilde{f} \equiv 0$. But then $\widetilde{f}'(0) = f'(0) = 0$, in contradiction to the fact that $s_0 > 0$. Thus f is injective, and we are done. □

H 3.4 (Local normal vs local uniform convergence of series)
Let $f_j : U \to \mathbb{C}$ be a sequence of complex-valued functions. The series $\sum_{j=1}^{\infty} f_j$ is called locally normally convergent if for each $z_0 \in U$ there exists $r > 0$ such that

$$\sum_{j=1}^{\infty} \sup_{z \in B_r(z_0)} |f_j(z)| < +\infty.$$

a) Show that if $\sum_{j=1}^{\infty} f_j$ is locally normally convergent then the sequence $g_n := \sum_{j=1}^{n} f_j$ is locally uniformly convergent.

b) Give an example of (not necessarily holomorphic) functions $f_j : \mathbb{C} \to \mathbb{C}$ which shows that the converse is in general false.

Solution H 3.4:
a) Local normal convergence implies in particular that the series $\sum_{j=1}^{\infty} f_j(z)$ converges absolutely, for all $z \in U$. Now fix $z_0 \in U$ and let $r > 0$ be given by the definition of local normal convergence. Then for any $z \in B_r(z_0)$ we have by the triangle inequality for series that

$$\sup_{z \in B_r(z_0)} \left| \sum_{j=1}^{\infty} f_j(z) - \sum_{j=1}^{n} f_j(z) \right| \le \sup_{z \in B_r(z_0)} \sum_{j=n+1}^{\infty} |f_j(z)| \le \sum_{j=n+1}^{\infty} \sup_{z \in B_r(z_0)} |f_j(z)|.$$

Since the sum $\sum_{j=1}^{\infty} \sup_{z \in B_r(z_0)} |f_j(z)|$ is finite by assumption, the right hand side of the above inequality converges to 0 as $n \to +\infty$. This proves the claim. □

b) In the above proof there are two estimates which are not sharp (and these we should use to produce a counterexample). While the first one was about the triangle inequality, the second interchanged sum and suprema. We exploit the first one and take sign-changing functions. Define $f_j : \mathbb{C} \to \mathbb{C}$ by $f_j(z) = \frac{(-1)^{j+1}}{j} e^z$. Since the alternating harmonic series converges to $\log(2)$ and the exponential function is bounded on each compact set $K \subset \mathbb{C}$, for every $z_0 \in \mathbb{C}$ we observe that

$$\sup_{z \in B_1(z_0)} \left| \log(2) e^z - \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j} e^z \right| \le e^{|z_0|+1} \left| \log(2) - \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j} \right| \to 0 \quad \text{as } n \to +\infty.$$

Hence the partial sums $g_n = \sum_{j=1}^{n} f_j$ converge locally uniformly to the function $g(z) = \log(2) e^z$. On the other hand, for $z_0 = 0$ we obtain for any $r > 0$ that

$$\sum_{j=1}^{\infty} \sup_{z \in B_r(0)} |f_j(z)| \ge \sum_{j=1}^{\infty} \frac{1}{j} = +\infty,$$

which shows that the series $\sum_{j=1}^{\infty} f_j$ is not locally normally convergent. □

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[3] One has to take some care here since $\widetilde{f}_n(z) := f_n(z) - f(z_1)$ could have a single **multiple** zero at $z = z_1$. This actually cannot happen, since injectivity of f_n implies that f_n' does not vanish (Lemma 7.1), but we do not know this yet. To get around this, let $B_1, B_2 \subset D$ be non-intersecting open balls containing z_1 and z_2, respectively. Then apply Corollary 1.6 to each ball, which implies that either $\widetilde{f} \equiv 0$ (on an open ball, hence in D by analytic continuation), or for every n sufficiently large $f_n(z) - f(z_1)$ must have a zero in each ball. This contradicts injectivity of f_n.